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Trajectory Shaping in Linear–Quadratic Pursuit–Evasion Games

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I. Introduction

INTERCEPTION of a maneuverable target can be formulated as a zero-sum pursuit–evasion game. For the sake of analytical solvability, there is interest in using a linear game model. Fortunately, in most cases the relative end-game trajectory can be linearized. The nonlinear phenomenon of bounded lateral acceleration can be modeled either by bounded controls or by adding quadratic penalty terms on the (assumed to be unbounded) controls to the square of the miss distance, the natural cost function of the game.[1,2]

The clear advantage of the linear–quadratic formulation is a continuous and smooth linear control strategy, in contrast to the discontinuous (bang–bang) control of the game with bounded controls, at the expense of some increase of the miss distance. In a recent paper,[3] it was shown that by decreasing the pursuer's penalty coefficient, while keeping its ratio to the penalty coefficient of the evader constant, the guaranteed homing accuracies of the linear–quadratic game solution and of the game with bounded controls become similar.

Another advantage of the linear–quadratic differential game (LQDG) formulation is its flexibility, which enables it not only to include in the cost function additional weights on other terminal variables, but also to introduce some "trajectory shaping" by augmenting the cost function with a running-cost (quadratic-integral) term on the state variables. In a very recent study[4] it was discovered that the trajectory-shaping term also leads to attenuation of the disturbance created by random maneuvering of the evader.

In this short Note the effect of the trajectory-shaping term on such disturbance attenuation is presented and analyzed. The model used for the analysis assumes planar geometry, first-order pursuer, and ideal evader dynamics. The analysis leads to a differential Riccati equation that needs to be solved. A simple technique for facilitating the solution is proposed.

The Note is organized as follows: In the next section the standard two-dimensional problem geometry and the mathematical modeling will be reviewed. The problem formulation and analysis by the LQDG theory will be presented in Sec. III. Section IV presents some numerical results including a comparison between the LQDG and the hard-bounded game results. Section V summarizes the paper.

II. Mathematical Modeling

We shall make the following assumptions:

1) The end game is two-dimensional and takes place in the horizontal plane (gravity is compensated for independently).

2) The speeds of the pursuer (the missile) P and the evader (the target) E are constant during the end game (approximately true for short end games).

3) The trajectories of P and E can be linearized around their collision course.

4) The pursuer is more maneuverable than the evader.

5) We assume first-order pursuer and ideal evader dynamics (conservative assumption from the pursuer's point of view).

6) The pursuer can measure its normal acceleration in addition to the relative separation and velocity, and it has an estimate of the time to go.

We assume that the collision condition is satisfied (Fig. 1); namely,

$$V_p \sin(\gamma_{p_0}) - V_e \sin(\gamma_{e_0}) = 0 \tag{1}$$

where V_e and V_p are the pursuer's and evader's velocities and $\gamma_{p_0}, \gamma_{e_0}$ are the pursuer's and evader's nominal heading angles, respectively. In this case, the nominal closing velocity V_c is given by

$$V_c = -\dot{R} = V_p \cos(\gamma_{p_0}) - V_e \cos(\gamma_{e_0}) \approx \text{const} \tag{2}$$

and the (nominal) terminal time is given by

$$t_f = R_0 / V_c \tag{3}$$

where R_0 is the initial length of the line of sight.

Let Y_e, Y_p be the separation (Fig. 1) of the evader and the pursuer, respectively, from the nominal line of sight, and let y be the relative separation (i.e., $y \equiv Y_e - Y_p$), leading to the dynamic equation

$$\dot{y} = \dot{Y}_e - \dot{Y}_p = V_e \sin(\gamma_{e_0} + \gamma_e) - V_p \sin(\gamma_{p_0} + \gamma_p) \tag{4}$$

where γ_p, γ_e are the deviations of the pursuer's and evader's headings from the nominal collision values, respectively. If these deviations are small enough, we may use an approximation to obtain

$$\sin(\gamma_{p_0} + \gamma_p) \approx \sin(\gamma_{p_0}) + \cos(\gamma_{p_0})\gamma_p \tag{5}$$

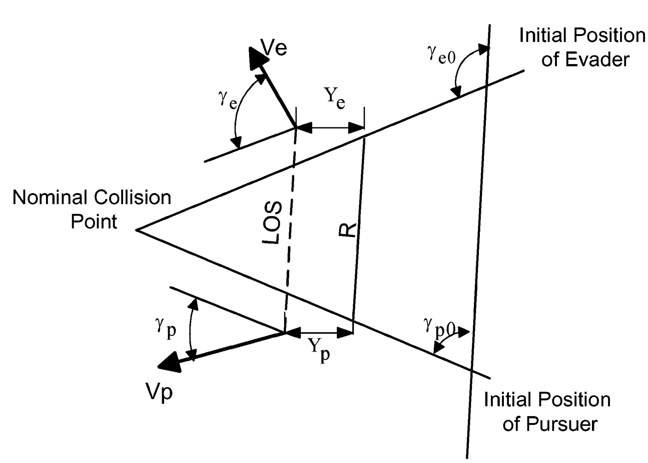

Fig. 1 Problem geometry.

Received 3 April 2004; revision received 6 June 2004; accepted for publication 7 June 2004.

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$$\sin(\gamma_{e_0} + \gamma_e) \approx \sin(\gamma_{e_0}) + \cos(\gamma_{e_0})\gamma_e \tag{6}$$

Substituting the results into Eq. (4), we arrive at

$$\dot{y} = \dot{Y}_e - \dot{Y}_p = V_e \cos(\gamma_{e_0})\gamma_e - V_p \cos(\gamma_{p_0})\gamma_p \tag{7}$$

III. Problem Formulation and Analysis

A. Problem Formulation

Define

$$x_1 = y, x_2 = \frac{\mathrm{d}y}{\mathrm{d}t}, \qquad x_3 = -V_p \cos(\gamma_{p_0}) \frac{\mathrm{d}\gamma_p}{\mathrm{d}t} \tag{8}$$

$$u = -V_p \cos(\gamma_{p_0}) \frac{\mathrm{d}\gamma_{pc}}{\mathrm{d}t}, \qquad w = V_e \cos(\gamma_{e_0}) \frac{\mathrm{d}\gamma_e}{\mathrm{d}t} \tag{9}$$

where x_3 is the pursuer's actual acceleration normal to the initial line of sight and u is the corresponding command, and where the target's actual acceleration normal to the initial line of sight is w.

The problem has the following state-space representation, with u being the control function and w a disturbance function:

$$\dot{x} = Ax + Bu + Dw$$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -\frac{1}{T} \end{bmatrix} B = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{T} \end{bmatrix} D = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \tag{10}$$

where T is the pursuer's time constant.

We formulate the following minimax problem:

$$\min_u \max_w J = \frac{b}{2} x_1^2(t_f) + \frac{1}{2} \int_0^{t_f} x^T Q x + u^2(t) - \gamma^2 w^2(t)\, \mathrm{d}t$$

$$\gamma > 0 \tag{11}$$

The positive coefficient γ^2 weights the evasive maneuvers. For disturbance attenuation purposes we want to let $\gamma \to \min$.

Q is a semi-positive-definite trajectory-shaping penalty matrix that penalizes the state deviation from the nominal collision path. For simplicity we restrict this Note to the following form:

$$Q = \begin{bmatrix} q & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{12}$$

Other forms of Q may also be of interest[4] but are beyond the scope of this Note.

B. Saddle-Point Solution

The theory of linear–quadratic differential games is covered in many textbooks (e.g., Ref. 1, Chap. 2, and Ref. 5, Chap. 4) and we shall not try to cover it here. For high enough γ the game has a saddle-point solution that can be obtained by solving the differential Riccati equation (DRE)

$$-\dot{P} = PA + A^T P - PBB^T P + \gamma^{-2} PDD^T P + Q \tag{13}$$

where the terminal conditions are

$$P(t_f) = \begin{bmatrix} b & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{14}$$

and the optimal pursuer and evader strategies are of the form (continuous function)

$$u^* = -B^T P x, \qquad w^* = \gamma^{-2} D^T P x \tag{15}$$

Notice that to minimize γ the condition for no conjugate point on the interval $[0, t_f]$ should be satisfied. For a given interval $[0, t_f]$ there exists a (minimal) γ_{cr} such that for $\gamma \leq \gamma_{\mathrm{cr}}$ the solution does exist.

C. Guidelines for Preliminary Estimation of γ_{cr}

To alleviate the inevitable search for γ_{cr}, it is worthwhile to limit the domain of search using available results. There are two limit cases for the cost given by Eq. (11), namely $q = 0$ and $b = 0$.

The first case, $q = 0$, has a closed-form solution. The condition for a nonconjugate point has been formulated explicitly in Ref. 1, Eq. (4.26). It states that the positive roots of the function

$$P(h) = 6/T^3 b + (1 - \gamma^{-2})h^3 + 3 - 6h^2 + 6h - 12he^{-h} - 3e^{-2h}) \tag{16}$$

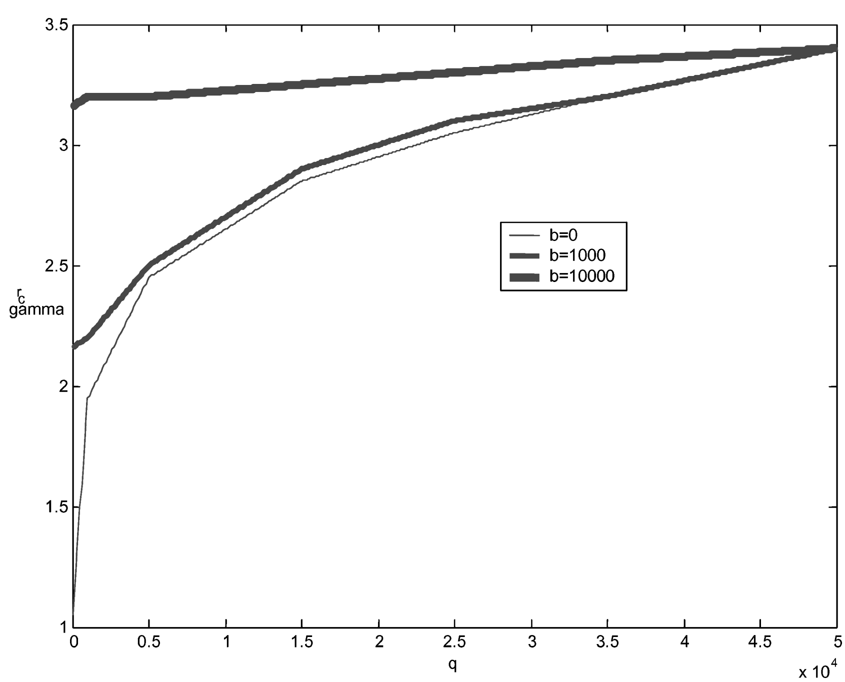

Fig. 2 Conjugate-point criteria.

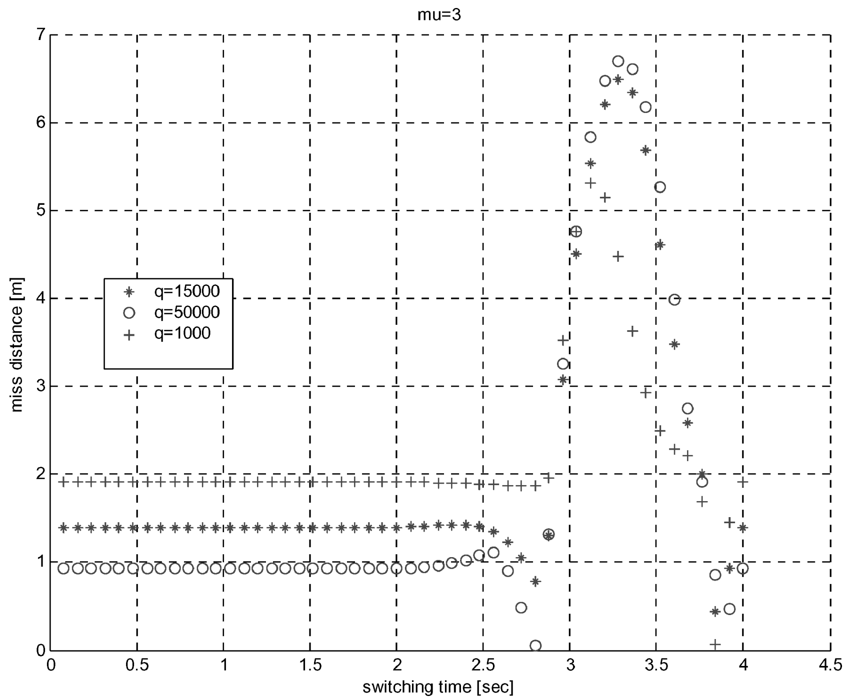

Fig. 3 LQDG performance.

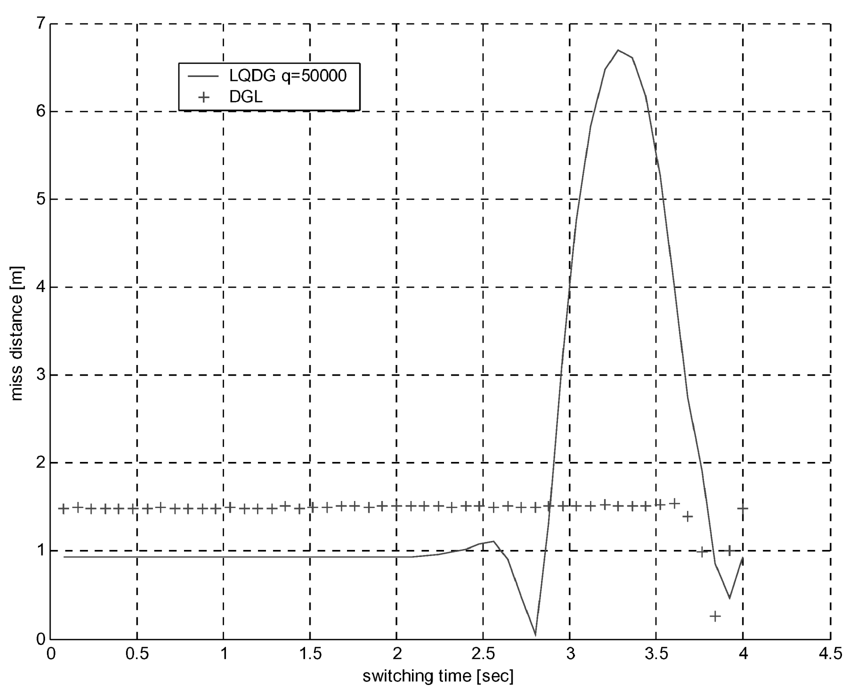

Fig. 4 Performance of two games.

should not lie in $[0, t_f/T]$ (the first occurrence inside the interval constitutes a conjugate point). Adding the trajectory-shaping term with positive q to the cost J would require a higher γ_{cr}

The second limit case results from Theorem 4.8 of Ref. 5, which states that the problem (12) and (13) with $b = 0$ has a solution if and only if the algebraic Riccati equation (ARE)

$$0 = PA + A^T P - PBB^T P + \gamma^{-2} PDD^T P + Q \qquad (17)$$

has a nonnegative definite solution.

[For the applicability of the theorem we require the detectability of the pair $(A, Q^{1/2})$, which can easily be verified in our problem.]

To illustrate this, consider an example with $T = 0.25$ s, $t_f = 4$ s. Figure 2 presents γ_{cr} as a function of q for three values of b, namely $\{0, 1000, 10{,}000\}$. The results for any given pair (b, q) were obtained by searching the values of γ where the solution for the DRE (13) ceases to exist (finite escape times). The intersection with the γ_{cr} axis (i.e., $q = 0$) is also the critical result of Eq. (16) when its positive roots lie in the game duration. The values for $b = 0$ (lower curve) coincide with the γ_{cr} values of the ARE [Eq. (17)], the minimal values with nonnegative solutions.

As expected, increasing b and/or q (positive definite terms in the cost) results with monotone increase of γ_{cr}. Notice the interesting phenomenon that for $q \gg b$, the γ_{cr} values approach the AREs.

These results can help us to estimate lower and upper bounds for γ_{cr}. For a given problem [formulated with a given pair (b, q)], we can first solve Eq. (16) to find the critical values for the case $q = 0$.

This will provide a lower bound for the problem. We then can solve the ARE with $\bar{q} \gg \max(q, b)$. This artificial weight is devised based on the asymptotic properties of the problem. The critical value of the ARE problem provides an estimate for the upper value of $\gamma_{\rm cr}$. Having obtained lower and upper estimates one should search in the relevant segment and solve the DRE [Eq. (13)] with varying γ until a finite escape time occurs within the game duration.

IV. Simulation Results

In this section, a numerical example that illustrates the merits of the trajectory-shaping-based guidance law is presented. We analyze the effect on the miss distance against a bang–bang acceleration target maneuver, which was found to be the optimal evasive maneuver under certain optimal-control and differential-game problems (e.g., Refs. 2, 6, and 7). We have assumed the same values as used in the previous example (i.e., $t_f = 4$ s, $T = 0.25$ s); hence the results of Fig. 2 apply. The lateral acceleration of the pursuer is limited to $15g$. The lateral acceleration of the target is limited to $7.5g$, giving the maneuverability ratio $\mu = 2$ (a realistic value in present-day combat scenarios). The switching point of the evasive maneuver varies in different simulation runs from the initial time to the point of closest approach of the end game. We set $b = 1000$ and consider various values for the nonnegative design parameter q. Because it is not advisable to work at (or very close to) the true conjugate value of Fig. 2, we use $\gamma = \gamma_{\rm cr} + 0.1$.

Figure 3 depicts the miss distance as a function of the trajectory-shaping weight. As q increases, smaller miss distances are obtained, up to a certain value of q for which the miss distance approaches a minimal value almost independent of the switch point in the major (initial) part of the end game. Larger miss distances are obtained only for a limited interval of switch points $t_{\rm go} = T$–$3T$.

Figure 4 compares the miss distances of the linear–quadratic game (LQDG) with the bounded-control game (DGL) of Ref. 2. The pursuer control in DGL is obtained by using the discontinuous control function[2]

$$u = u_{\max}\,\mathrm{sign}[Z(t_{\rm go})], \qquad t_{\rm go} = t_f - t \tag{18}$$

where Z is the zero-effort miss distance obtained by

$$Z = x_1 + x_2 t_{\rm go} - T^2 x_3[e^{-\theta} + \theta - 1], \qquad \theta = t_{\rm go}/T \tag{19}$$

For most cases the results of LQDG are slightly better. Only for evasive maneuvers that take place in the time frame $t_{\rm go} = T$–$3T$ is the performance of the bounded-control game solution superior. Assuming a uniformly distributed $t_{\rm switch}$, between 0 and 4 s, we get average miss distances of 1.4 and 1.6 m for LQDG and DGL, respectively. A heuristic explanation for the contribution of the trajectory-shaping term is as follows. The classical linear–quadratic game[1] avoids maneuvering at early stages because the integral of the control term is negatively affected and deferring the maneuver is profitable. It trades off early control effort for terminal miss. Adding the new term forces the missile to react earlier to evasive maneuvers at the expense of a larger control effort, in order to remain closer to the collision course. This in fact is the underlying philosophy of the hard-bound differential-game approach that counteracts the instantaneous zero-effort miss.

V. Conclusions

The disturbance-attenuation effect of the trajectory-shaping term in linear–quadratic differential games was presented and analyzed. The solution involves a search over the possible solution domain of the associated differential Riccati equation. A contribution of this Note is to employ available results to limit the domain of search, thus simplifying the computation.

However, the main contribution of this research is in showing that by increasing the weights on the trajectory-shaping term we reduce the miss distances. The effect becomes saturated at a certain limit value. The results obtained by applying this limit are compared with the miss distances achieved by the solution of the hard-bound differential game, which leads to a highly discontinuous control. Under the present formulation we obtain similar results with a smooth controller.

Although the hard-bound differential game strategy is superior in worst-case scenarios (where sophisticated and smart targets can execute precisely timed evasive maneuvers), against randomly maneuvering targets the new guidance law has similar performance in terms of average miss distances.

In summary, to improve the performance of the classical linear–quadratic game solution against stressing target maneuvers this preliminary study advocates using for guidance law synthesis the linear–quadratic game formulation with the inclusion of a trajectory-shaping term in the cost function.

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Earth Escape by Ideal Sail and Solar-Photon Thrustor Spacecraft

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Introduction

SOLAR sails use the solar radiation pressure on a large reflecting surface to obtain low-thrust propulsion. This technology has been identified as enabling many recent space mission concepts. An interesting application involves the study of escape trajectories from the Earth. Early contributions to this subject date back to Sands[1] and Fimple,[2] who considered initial circular orbits and used other simplifying assumptions, and to Sackett and Edelbaum.[3] Locally optimal steering laws for a flat sail have been considered in various forms by different authors.[4–6] In a recent paper Coverstone and Prussing[7] investigated the problem of Earth escape from a geosynchronous transfer orbit with an ideal flat sail through a sail-force control algorithm that maximizes the instantaneous rate of increase of the total orbital energy. In their analysis a spherical gravity model for the Earth is assumed, and only the solar gravitational perturbation is included.

Received 4 May 2004; revision received 6 July 2004; accepted for publication 23 July 2004.

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